

THE DIRAC OPERATOR IN A FERMION BAG BACKGROUND IN 1 + 1 DIMENSIONS AND GENERALIZED SUPERSYMMETRIC QUANTUM MECHANICS

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Abstract

We show that the spectral theory of the Dirac operator $D = i\cancel{\partial} - \sigma(x) - i\pi(x)\gamma_5$ in a static background $(\sigma(x), \pi(x))$ in 1 + 1 space-time dimensions, is underlined by a certain generalization of supersymmetric quantum mechanics, and explore its consequences.

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1 Introduction: Bags and Resolvents

A central concept in particle physics states that fundamental particles acquire their masses through interactions with vacuum condensates. Thus, a massive particle may carve out around itself a spherical region [1] or a shell [2] in which the condensate is suppressed, thus reducing the effective mass of the particle at the expense of volume and gradient energy associated with the condensate. This picture has interesting phenomenological consequences [1, 3].

This phenomenon may be studied non-perturbatively in model field theories in $1 + 1$ space-time dimensions such as the Gross-Neveu (GN) model [4] and the multi-flavor Nambu-Jona-Lasinio (NJL) [5] model, in the large N limit.

Explicit calculations of fermion bag profiles in the GN and NJL models were given originally in [6], [7] and in [8].

Following these works, fermion bags in the GN and NJL models were discussed in the literature several other times [9], [10], [11], [12]. For a recent review on these and related matters, see [13].

Very recently, static chiral fermion bag solitons [14] in a $1 + 1$ dimensional model, as well as non-chiral (real scalar) fermion bag solitons [15], were discussed, in which the scalar field that couples to the fermions was dynamical already at the classical level.

Mathematical considerations similar to those involved in studying fermion bags, appear also in other branches of theoretical physics, such as the theory of inhomogeneous superconductors [16], and the results of this paper may be applicable there as well.

Studying the physics of fermion bags necessarily involves knowledge of the resolvent of the Dirac operator in the background of the bag. As an example, let us consider the $1 + 1$ dimensional NJL model (which contains the GN model as a special case).

One version of writing the action of the 1 + 1 dimensional NJL model is

$$S = \int d^2x \left\{ \sum_{a=1}^N \bar{\psi}_a \left[i\rlap{\not{\partial}} - (\sigma + i\pi\gamma_5) \right] \psi_a - \frac{1}{2g^2} (\sigma^2 + \pi^2) \right\}, \quad (1.1)$$

where the ψ_a ($a = 1, \dots, N$) are N flavors of massless Dirac fermions, with Yukawa couplings to the scalar and pseudoscalar auxiliary fields $\sigma(x), \pi(x)$.

The partition function associated with (1.1) is

$$\mathcal{Z} = \int \mathcal{D}\sigma \mathcal{D}\pi \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp i \int d^2x \left\{ \bar{\psi} \left[i\rlap{\not{\partial}} - (\sigma + i\pi\gamma_5) \right] \psi - \frac{1}{2g^2} (\sigma^2 + \pi^2) \right\} \quad (1.2)$$

Integrating over the grassmannian variables leads to $\mathcal{Z} = \int \mathcal{D}\sigma \mathcal{D}\pi \exp\{iS_{eff}[\sigma, \pi]\}$ where the bare effective action is

$$S_{eff}[\sigma, \pi] = -\frac{1}{2g^2} \int d^2x (\sigma^2 + \pi^2) - iN \text{Tr} \log \left[i\rlap{\not{\partial}} - (\sigma + i\pi\gamma_5) \right] \quad (1.3)$$

and the trace is taken over both functional and Dirac indices.

This theory has been studied in the limit $N \rightarrow \infty$ with Ng^2 held fixed[4]. In this limit, (1.2) is governed by saddle points of (1.3) and the small fluctuations around them. The most general saddle point condition reads

$$\begin{aligned} \frac{\delta S_{eff}}{\delta \sigma(x, t)} &= -\frac{\sigma(x, t)}{g^2} + iN \text{tr} \left[\langle x, t | \frac{1}{i\rlap{\not{\partial}} - (\sigma + i\pi\gamma_5)} | x, t \rangle \right] = 0 \\ \frac{\delta S_{eff}}{\delta \pi(x, t)} &= -\frac{\pi(x, t)}{g^2} - N \text{tr} \left[\gamma_5 \langle x, t | \frac{1}{i\rlap{\not{\partial}} - (\sigma + i\pi\gamma_5)} | x, t \rangle \right] = 0. \end{aligned} \quad (1.4)$$

Fermion bags are the space-time dependent solutions $(\sigma(x, t), \pi(x, t))$ of (1.4), subjected to appropriate boundary conditions at spatial infinity, and on which S_{eff}/N is finite.

Thus, studying fermion bags necessarily involves the resolvent of the Dirac operator in the background of the bag.

In this paper we discuss some mathematical aspects of the much simpler problem of *static* fermion bags, namely, the static solutions $(\sigma(x), \pi(x))$ of (1.4).

For the usual physical reasons, we set boundary conditions on our static fields such that $\sigma(x)$ and $\pi(x)$ start from a point on the vacuum manifold $\sigma^2 + \pi^2 = m^2$ (with constant σ, π of course, and where m is the dynamical mass [4]) at $x = -\infty$, wander around in the $\sigma - \pi$ plane, and then relax back to another point on the vacuum manifold at $x = +\infty$. Thus, we must have the asymptotic behavior

$$\begin{aligned} \sigma &\xrightarrow{x \rightarrow \pm\infty} m \cos \theta_{\pm} \quad , \quad \sigma' \xrightarrow{x \rightarrow \pm\infty} 0 \\ \pi &\xrightarrow{x \rightarrow \pm\infty} m \sin \theta_{\pm} \quad , \quad \pi' \xrightarrow{x \rightarrow \pm\infty} 0 \end{aligned} \quad (1.5)$$

where θ_{\pm} are the asymptotic chiral alignment angles. Only the difference $\theta_+ - \theta_-$ is meaningful, of course, and henceforth we use the axial $U(1)$ symmetry of (1.1) to set $\theta_- = 0$, such that $\sigma(-\infty) = m$ and $\pi(-\infty) = 0$. We also omit the subscript from θ_+ and denote it simply by θ from now on. As typical of solitonic configurations, we expect, that $\sigma(x)$ and $\pi(x)$ tend to their asymptotic boundary values (1.5) on the vacuum manifold at an exponential rate which is determined, essentially, by the mass gap m of the model.

Thus, in order to study static fermion bags, we need to invert the Dirac operator

$$D \equiv i\not{\partial} - (\sigma(x) + i\pi(x)\gamma_5) \quad (1.6)$$

in a given background of static field configurations $\sigma(x)$ and $\pi(x)$, subjected to the boundary conditions (1.5). In particular, we have to find the diagonal resolvent of (1.6) in that background. We stress that inverting (1.6) has nothing to do with the large N approximation, and consequently our results are valid for any value of N .

The rest of the paper is organized as follows: In Section 2 we show that the Dirac equation $(i\not{\partial} - (\sigma(x) + i\pi(x)\gamma_5))\psi = 0$ in a given static $\sigma(x) + i\gamma_5\pi(x)$ background, is equivalent to a pair of two isospectral Sturm-Liouville equations in one dimension, which generalize the well known one-dimensional supersymmetric quantum mechanics. We use this generalized supersymmetry to express all four entries of the space-diagonal Dirac resolvent (i.e., the resolvent evaluated at coincident spatial

coordinates) in terms of a single function. In Section 3, we use the results of Section 2 to derive simple expressions for various bilinear fermion condensates in the given static $\sigma(x) + i\gamma_5\pi(x)$ background. In particular, we prove that each frequency mode of the spatial current $\langle\bar{\psi}(x)\gamma^1\psi(x)\rangle$ vanishes identically in the static background.

2 Resolvent of the Dirac Operator With Static Background Fields

As was explained in the introduction, we need to invert the Dirac operator (1.6), $D \equiv i\cancel{D} - (\sigma(x) + i\pi(x)\gamma_5)$, in a given background of static field configurations $\sigma(x)$ and $\pi(x)$, subjected to the boundary conditions (1.5).

In this paper we use the Majorana representation

$$\gamma^0 = \sigma_2, \quad \gamma^1 = i\sigma_3 \quad \text{and} \quad \gamma^5 = -\gamma^0\gamma^1 = \sigma_1 \quad (2.1)$$

for γ matrices. In this representation (1.6) becomes

$$D = \begin{pmatrix} -\partial_x - \sigma & -i\omega - i\pi \\ i\omega - i\pi & \partial_x - \sigma \end{pmatrix} = \begin{pmatrix} -Q & -i\omega - i\pi \\ i\omega - i\pi & -Q^\dagger \end{pmatrix}, \quad (2.2)$$

where we introduced the pair of adjoint operators

$$Q = \sigma(x) + \partial_x, \quad Q^\dagger = \sigma(x) - \partial_x. \quad (2.3)$$

(To obtain (2.2), we have naturally transformed $i\cancel{D} - (\sigma(x) + i\pi(x)\gamma_5)$ to the ω plane, since the background fields $\sigma(x), \pi(x)$ are static.)

Inverting (2.2) is achieved by solving

$$\begin{pmatrix} -Q & -i\omega - i\pi(x) \\ i\omega - i\pi(x) & -Q^\dagger \end{pmatrix} \cdot \begin{pmatrix} a(x, y) & b(x, y) \\ c(x, y) & d(x, y) \end{pmatrix} = -i\mathbf{1}\delta(x - y) \quad (2.4)$$

for the Green's function of (2.2) in a given background $\sigma(x), \pi(x)$. By dimensional analysis, we see that the quantities a, b, c and d are dimensionless.

2.1 Generalized ‘‘Supersymmetry’’ in a Chiral Bag Background

We now show that the spectral theory of the Dirac operator (2.2) is underlined by a certain generalized one dimensional supersymmetric quantum mechanics. This generalized supersymmetry is very helpful in simplifying various calculations involving the Dirac operator and its resolvent.

The diagonal elements $a(x, y)$, $d(x, y)$ in (2.4) may be expressed in term of the off-diagonal elements as

$$a(x, y) = \frac{-i}{\omega - \pi(x)} Q^\dagger c(x, y), \quad d(x, y) = \frac{i}{\omega + \pi(x)} Q b(x, y) \quad (2.5)$$

which in turn satisfy the second order partial differential equations

$$\begin{aligned} & \left[Q^\dagger \frac{1}{\omega + \pi(x)} Q - (\omega - \pi(x)) \right] b(x, y) = \\ & -\partial_x \left[\frac{\partial_x b(x, y)}{\omega + \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] \frac{b(x, y)}{\omega + \pi(x)} = \delta(x - y) \\ & \left[Q \frac{1}{\omega - \pi(x)} Q^\dagger - (\omega + \pi(x)) \right] c(x, y) = \\ & -\partial_x \left[\frac{\partial_x c(x, y)}{\omega - \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 + \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega - \pi(x)} \right] \frac{c(x, y)}{\omega - \pi(x)} = -\delta(x - y). \end{aligned} \quad (2.6)$$

Thus, $b(x, y)$ and $-c(x, y)$ are simply the Green's functions of the corresponding second order Sturm-Liouville operators¹

$$\begin{aligned} L_b(\omega)b(x) &= -\partial_x \left[\frac{\partial_x b(x)}{\omega + \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 - \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega + \pi(x)} \right] \frac{b(x)}{\omega + \pi(x)} \\ L_c(\omega)c(x) &= -\partial_x \left[\frac{\partial_x c(x)}{\omega - \pi(x)} \right] + \left[\sigma(x)^2 + \pi(x)^2 + \sigma'(x) - \omega^2 + \frac{\sigma(x)\pi'(x)}{\omega - \pi(x)} \right] \frac{c(x)}{\omega - \pi(x)} \end{aligned} \quad (2.7)$$

in (2.6), namely,

$$b(x, y) = \frac{\theta(x - y) b_2(x) b_1(y) + \theta(y - x) b_2(y) b_1(x)}{W_b}$$

¹Note that ω plays here a dual role: in addition to its role as the spectral parameter (the ω^2 terms in (2.7)), it also appears as a parameter in the definition of these operators-hence the explicit ω dependence in our notations for these operators in (2.7). For this reason, it is possible to completely factorize the operators L_b and L_c by additional obvious ω -dependent similarity transformations on Q and Q^\dagger . However, these similarity transformations are singular at points where $\pi(x) = \pm\omega$ and are thus ill defined, and we will avoid them.

$$c(x, y) = -\frac{\theta(x-y)c_2(x)c_1(y) + \theta(y-x)c_2(y)c_1(x)}{W_c}. \quad (2.8)$$

Here $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$ are pairs of independent fundamental solutions of the two equations $L_b b(x) = 0$ and $L_c c(x) = 0$, subjected to the boundary conditions

$$b_1(x), c_1(x) \xrightarrow{x \rightarrow -\infty} A_{b,c}^{(1)}(k)e^{-ikx}, \quad b_2(x), c_2(x) \xrightarrow{x \rightarrow +\infty} A_{b,c}^{(2)}(k)e^{ikx} \quad (2.9)$$

with some possibly k dependent coefficients $A_{b,c}^{(1)}(k), A_{b,c}^{(2)}(k)$ and with²

$$k = \sqrt{\omega^2 - m^2}, \quad \text{Im}k \geq 0. \quad (2.10)$$

The purpose of introducing the (yet unspecified) coefficients $A_{b,c}^{(1)}(k), A_{b,c}^{(2)}(k)$ will become clear following Eqs. (2.13) and (2.14). The boundary conditions (2.9) are consistent, of course, with the asymptotic behavior (1.5) of σ and π due to which both L_b and L_c tend to a free particle hamiltonian $[-\partial_x^2 + m^2 - \omega^2]$ as $x \rightarrow \pm\infty$.

The wronskians of these pairs of solutions are

$$W_b(k) = \frac{b_2(x)b_1'(x) - b_1(x)b_2'(x)}{\omega + \pi(x)}$$

$$W_c(k) = \frac{c_2(x)c_1'(x) - c_1(x)c_2'(x)}{\omega - \pi(x)} \quad (2.11)$$

As is well known, $W_b(k)$ and $W_c(k)$ are independent of x .

Note in passing that the canonical asymptotic behavior assumed in the scattering theory of the operators L_b and L_c corresponds to setting $A_{b,c}^{(1)} = A_{b,c}^{(2)} = 1$ in (2.9). Thus, the wronskians in (2.11) are *not* the canonical wronskians used in scattering theory. As is well known in the literature [17], the *canonical* wronskians are proportional (with a k independent coefficient) to $k/t(k)$, where $t(k)$ is the transmission amplitude of the corresponding operator L_b or L_c . Thus, on top of the well-known

²We see that if $\text{Im}k > 0$, b_1 and c_1 decay exponentially to the left, and b_2 and c_2 decay to the right. Thus, if $\text{Im}k > 0$, both $b(x, y)$ and $c(x, y)$ decay as $|x - y|$ tends to infinity.

features of $t(k)$, such as the fact that $t(k)$ has simple poles on the positive imaginary k -axis (corresponding to bound states), the wronskians in (2.11) will have additional spurious k -dependence coming from the amplitudes $A_{b,c}^{(1)}(k), A_{b,c}^{(2)}(k)$ in (2.9).

Substituting the expressions (2.8) for the off-diagonal entries $b(x, y)$ and $c(x, y)$ into (2.5), we obtain the appropriate expressions for the diagonal entries $a(x, y)$ and $d(x, y)$. We do not bother to write these expressions here. It is useful however to note, that despite the ∂_x 's in the Q operators in (2.5), that act on the step functions in (2.8), neither $a(x, y)$ nor $d(x, y)$ contain pieces proportional to $\delta(x - y)$. Such pieces cancel one another due to the symmetry of (2.8) under $x \leftrightarrow y$.

We will now prove that the spectra of the operators L_b and L_c are essentially the same. Our proof is based on the fact that we can factorize the eigenvalue equations $L_b b(x) = 0$ and $L_c c(x) = 0$ as

$$\begin{aligned} \frac{1}{\omega - \pi(x)} Q^\dagger \frac{1}{\omega + \pi(x)} Q b &= b \\ \frac{1}{\omega + \pi(x)} Q \frac{1}{\omega - \pi(x)} Q^\dagger c &= c, \end{aligned} \tag{2.12}$$

as should be clear from (2.6) and (2.7).

The factorized equations (2.12) suggest the following map between their solutions. Indeed, given that $L_b b(x) = 0$, then clearly

$$c(x) = \frac{1}{\omega + \pi(x)} Q b(x) \tag{2.13}$$

is a solution of $L_c c(x) = 0$. Similarly, if $L_c c(x) = 0$, then

$$b(x) = \frac{1}{\omega - \pi(x)} Q^\dagger c(x) \tag{2.14}$$

solves $L_b b(x) = 0$.

Thus, in particular, given a pair $\{b_1(x), b_2(x)\}$ of independent fundamental solutions of $L_b b(x) = 0$, we can obtain from it a pair $\{c_1(x), c_2(x)\}$ of independent fundamental solutions of $L_c c(x) = 0$ by using (2.13), and vice versa. Therefore, with

no loss of generality, we henceforth assume, that the two pairs of independent fundamental solutions $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$, are related by (2.13) and (2.14).

The coefficients $A_{b,c}^{(1)}(k), A_{b,c}^{(2)}(k)$ in (2.9) are to be adjusted according to (2.13) and (2.14), and this was the purpose of introducing them in the first place.

Thus, with no loss of generality, we may make the standard choice

$$A_b^{(1)} = A_b^{(2)} = 1 \quad (2.15)$$

in (2.9). The coefficients $A_c^{(1)}, A_c^{(2)}$ are then determined by (2.13):

$$\begin{aligned} A_c^{(1)} &= \frac{\sigma(-\infty) - ik}{\pi(-\infty) + \omega} \\ A_c^{(2)} &= \frac{\sigma(\infty) + ik}{\pi(\infty) + \omega}. \end{aligned} \quad (2.16)$$

We note that these $b(x) \leftrightarrow c(x)$ mappings can break only if

$$Qb = 0 \quad \text{or} \quad Q^\dagger c = 0, \quad (2.17)$$

for $b(x)$ or $c(x)$ that *solve* (2.12). Do such solutions exist? Let us assume, for example, that $Qb = 0$ and that $L_b b = 0$. From the first equation in (2.12) (or in (2.6)), we see that this is possible if and only if $\omega \pm \pi(x) \equiv 0$, which clearly cannot hold if $\partial_x \pi(x) \neq 0$. A similar argument holds for $Q^\dagger c = 0$. Thus, if $\partial_x \pi(x) \neq 0$, the mappings (2.13) and (2.14) are one-to-one. In particular, a bound state in L_b implies a bound state in L_c (at the same energy) and vice-versa.

An interesting related result concerns the wronskians W_b and W_c . From (2.11), and from (2.13) and (2.14) it follows immediately that for pairs of independent fundamental solutions $\{b_1(x), b_2(x)\}$ and $\{c_1(x), c_2(x)\}$ we have

$$W_c = \frac{c_2 \partial_x c_1 - c_1 \partial_x c_2}{\omega - \pi(x)} = c_1 b_2 - c_2 b_1 = \frac{b_2 \partial_x b_1 - b_1 \partial_x b_2}{\omega + \pi(x)} = W_b. \quad (2.18)$$

The wronskians of pairs of independent fundamental solutions of L_b and L_c , which are related via (2.13) and (2.14) are equal!

To summarize, if $\partial_x \pi(x) \neq 0$, L_b and L_c have the same set of energy eigenvalues and their eigenfunctions are in one-to-one correspondence.

If, however, $\pi = \text{const.}$, then we are back to the familiar “supersymmetric” factorization

$$Q^\dagger Q b = (\omega^2 - \pi^2) b, \quad Q Q^\dagger c = (\omega^2 - \pi^2) c, \quad (2.19)$$

and mappings

$$c(x) = \frac{1}{\omega + \pi} Q b(x), \quad b(x) = \frac{1}{\omega - \pi} Q^\dagger c(x). \quad (2.20)$$

As is well known from the literature on supersymmetric quantum mechanics, the mappings (2.20) break down if either $Qb = 0$ or $Q^\dagger c = 0$, in which case the two operators $Q^\dagger Q$ and $Q Q^\dagger$ are isospectral, but only up to a “zero-mode” (or rather, an $\omega^2 = \pi^2$ mode), which belongs to the spectrum of only one of the operators³. The case $\pi(x) \equiv 0$ brings us back to the GN model. Supersymmetric quantum mechanical considerations were quite useful in the study of fermion bags in [10].

The “Witten index” associated with the pair of isospectral operators L_b and L_c , is always null for backgrounds in which $\partial_x \pi(x) \neq 0$, since they are absolutely isospectral, and not only up to zero modes. There is no interesting topology associated with spectral mismatches of L_b and L_c . This is not surprising at all, since the NJL model, with its continuous axial symmetry, does not support topological solitons. This is in contrast to the GN model, for which $\pi \equiv 0$, which contains topological kinks, whose topological charge is essentially the Witten index of the pair of operators (2.19).

We note in passing that isospectrality of L_b and L_c which we have just proved, is consistent with the γ_5 symmetry of the system of equations in (2.4), which relates the resolvent of D with that of $\tilde{D} = -\gamma_5 D \gamma_5$. Due to this symmetry, we can map the pair of equations $L_b b(x, y) = \delta(x - y)$ and $L_c c(x, y) = -\delta(x - y)$ (Eqs. (2.6)) on each other by

$$b(x, y) \leftrightarrow -c(x, y) \quad \text{together with} \quad (\sigma, \pi) \rightarrow (-\sigma, -\pi). \quad (2.21)$$

(Note that under these reflections we also have $a(x, y) \leftrightarrow -d(x, y)$, as we can see from (2.5).) The reflection $(\sigma, \pi) \rightarrow (-\sigma, -\pi)$ just shifts both asymptotic chiral angles θ_\pm

³This is true for short range decaying potentials on the whole real line. Strictly speaking, (to the best of our knowledge) only the case $\pi = 0$ appears in the literature on supersymmetric quantum mechanics.

by the same amount π , and clearly does not change the physics. Since this reflection interchanges $b(x, y)$ and $c(x, y)$ without affecting the physics, these two objects must have the same singularities as functions of ω , consistent with isospectrality of L_b and L_c .

2.2 The Diagonal Resolvent

Following [18, 11] we define the diagonal resolvent $\langle x | iD^{-1} | x \rangle$ symmetrically as

$$\begin{aligned} \langle x | -iD^{-1} | x \rangle &\equiv \begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} \\ &= \frac{1}{2} \lim_{\epsilon \rightarrow 0^+} \begin{pmatrix} a(x, y) + a(y, x) & b(x, y) + b(y, x) \\ c(x, y) + c(y, x) & d(x, y) + d(y, x) \end{pmatrix}_{y=x+\epsilon} \end{aligned} \quad (2.22)$$

Here $A(x)$ through $D(x)$ stand for the entries of the diagonal resolvent, which following (2.5) and (2.8) have the compact representation⁴

$$\begin{aligned} B(x) &= \frac{b_1(x)b_2(x)}{W_b} \quad , \quad D(x) = \frac{i}{2} \frac{[\partial_x + 2\sigma(x)] B(x)}{\omega + \pi(x)} \quad , \\ C(x) &= -\frac{c_1(x)c_2(x)}{W_c} \quad , \quad A(x) = \frac{i}{2} \frac{[\partial_x - 2\sigma(x)] C(x)}{\omega - \pi(x)} \quad . \end{aligned} \quad (2.23)$$

We now use the generalized “supersymmetry” of the Dirac operator, which we discussed in the previous subsection, to deduce some important properties of the functions $A(x)$ through $D(x)$.

From (2.23) and from (2.3) we we have

$$A(x) = \frac{i}{2} \frac{\partial_x - 2\sigma(x)}{\omega - \pi(x)} \left(-\frac{c_1 c_2}{W_c} \right) = \frac{i}{2W_c} \frac{c_2 Q^\dagger c_1 + c_1 Q^\dagger c_2}{\omega - \pi(x)}.$$

Using (2.14) first, and then (2.13), we rewrite this expression as

$$A(x) = \frac{i}{2W_c} (c_2 b_1 + c_1 b_2) = \frac{i}{2W_c} \frac{b_1 Q b_2 + b_2 Q b_1}{\omega + \pi(x)}.$$

⁴ A, B, C and D are obviously functions of ω as well. For notational simplicity we suppress their explicit ω dependence.

Then, using the fact that $W_c = W_b$ (Eq. (2.18)) and (2.23), we rewrite the last expression as

$$A(x) = \frac{i}{2} \frac{\partial_x + 2\sigma}{\omega + \pi(x)} \left(\frac{b_1 b_2}{W_b} \right) = \frac{i}{2} \frac{(\partial_x + 2\sigma)B(x)}{\omega + \pi(x)}.$$

Thus, finally,

$$A(x) = D(x). \quad (2.24)$$

Supersymmetry renders the diagonal elements A and D equal.

Due to (2.23), $A = D$ is also a first order differential equation relating B and C . We can also relate the off diagonal elements B and C to each other more directly. From (2.23) and from (2.13) we find

$$C(x) = -\frac{c_1 c_2}{W_c} = -\frac{(Qb_1)(Qb_2)}{(\omega + \pi)^2 W_c}. \quad (2.25)$$

After some algebra, and using (2.18), we can rewrite this as

$$-(\omega + \pi)^2 C = \sigma^2 B + \sigma B' + \frac{b'_1 b'_2}{W_b}$$

The combination $b'_1 b'_2 / W_b$ appears in $B'' = (b_1 b_2 / W_b)''$. After using $L_b b_{1,2} = 0$ to eliminate b'_1 and b'_2 from B'' , we find

$$\frac{b'_1 b'_2}{W_b} = \frac{1}{2} B'' - \frac{\pi' B'}{2(\omega + \pi)} - \left(\sigma^2 + \pi^2 - \sigma' - \omega^2 + \frac{\sigma \pi'}{\omega + \pi} \right) B$$

Thus, finally, we have

$$-(\omega + \pi)^2 C = \frac{1}{2} B'' + \left(\sigma - \frac{\pi'}{2(\omega + \pi)} \right) B' - \left(\pi^2 - \sigma' - \omega^2 + \frac{\sigma \pi'}{\omega + \pi} \right) B. \quad (2.26)$$

In a similar manner we can prove that

$$(\omega - \pi)^2 B = -\frac{1}{2} C'' + \left(\sigma - \frac{\pi'}{2(\omega - \pi)} \right) C' + \left(\pi^2 + \sigma' - \omega^2 + \frac{\sigma \pi'}{\omega - \pi} \right) C. \quad (2.27)$$

We can simplify (2.26) and (2.27) further. After some algebra, and using (2.23) we arrive at

$$\begin{aligned} C(x) &= \frac{i}{\omega + \pi(x)} \partial_x D(x) - \frac{\omega - \pi(x)}{\omega + \pi(x)} B(x) \\ B(x) &= \frac{i}{\omega - \pi(x)} \partial_x A(x) - \frac{\omega + \pi(x)}{\omega - \pi(x)} C(x). \end{aligned} \quad (2.28)$$

Supersymmetry, namely, isospectrality of L_b and L_c , enables us to relate the diagonal resolvents of these operators, B and C , to each other.

Thus, we can use (2.23), (2.24) and (2.28) to eliminate three of the entries of the diagonal resolvent in (2.23), in terms of the fourth.

Note that the two relations in (2.28) transform into each other under

$$B \leftrightarrow -C \quad \text{simultaneously with} \quad (\sigma, \pi) \rightarrow (-\sigma, -\pi), \quad (2.29)$$

in consistency with (2.21). The relations in (2.28) are linear and homogeneous, with coefficients that for $\partial_x \pi(x) \neq 0$ do not introduce additional singularities in the ω plane. Thus, we see, once more, that B and C have the same singularities in the ω plane. We refer the reader to Section 4 in [11] for concrete examples of such resolvents.

The case $\pi(x) \equiv 0$ brings us back to the GN model. In the GN model, our B and C , coincide, respectively, with ωR_- and $-\omega R_+$, defined in Eqs. (9) and (10) in [10]. With these identifications, the relation $A = D$ (Eq. (2.24)) coincides essentially with Eq. (18) of [10]. The relations (2.26) and (2.27) were not discussed in [10], but one can verify them, for example, for the resolvents corresponding to the kink case $\sigma(x) = m \tanh mx$ (Eq. (29) in [10]), for which

$$C = -\frac{\omega}{2\sqrt{m^2 - \omega^2}}, \quad B = \left[\left(\frac{m \operatorname{sech} mx}{\omega} \right)^2 - 1 \right] C.$$

3 Bilinear Fermion Condensates and Vanishing of the Spatial Fermion Current

Following basic principles of quantum field theory, we may write the most generic flavor-singlet bilinear fermion condensate in our static background as

$$\begin{aligned} \langle \bar{\psi}_{a\alpha}(t, x) \Gamma_{\alpha\beta} \psi_{a\beta}(t, x) \rangle_{\text{reg}} &= N \int \frac{d\omega}{2\pi} \text{tr} \left[\Gamma \langle x | \frac{-i}{\omega \gamma^0 + i \gamma^1 \partial_x - (\sigma + i \pi \gamma_5)} | x \rangle_{\text{reg}} \right] \\ &= N \int \frac{d\omega}{2\pi} \text{tr} \left\{ \Gamma \left[\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} - \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{VAC}} \right] \right\}, \end{aligned} \quad (3.1)$$

where we have used (2.22). Here $a = 1, \dots, N$ is a flavor index, and the trace is taken over Dirac indices α, β . As usual, we regularized this condensate by subtracting from it a short distance divergent piece embodied here by the diagonal resolvent

$$\langle x | -iD^{-1} | x \rangle_{\text{VAC}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\text{VAC}} = \frac{1}{2\sqrt{m^2 - \omega^2}} \begin{pmatrix} i m \cos \theta & \omega + m \sin \theta \\ -\omega + m \sin \theta & i m \cos \theta \end{pmatrix} \quad (3.2)$$

of the Dirac operator in a vacuum configuration $\sigma_{\text{VAC}} = m \cos \theta$ and $\pi_{\text{VAC}} = m \sin \theta$.

In our convention for γ matrices (2.1) we have

$$\begin{pmatrix} A(x) & B(x) \\ C(x) & D(x) \end{pmatrix} = \frac{A(x) + D(x)}{2} \mathbf{1} + \frac{A(x) - D(x)}{2i} \gamma^1 + i \frac{B(x) - C(x)}{2} \gamma^0 + \frac{B(x) + C(x)}{2} \gamma_5. \quad (3.3)$$

An important condensate is the expectation value of the fermion current $\langle j^\mu(x) \rangle$. In particular, consider its spatial component. In our static background $(\sigma(x), \pi(x))$, it must, of course, vanish identically

$$\langle j^1(x) \rangle = 0. \quad (3.4)$$

Thus, substituting $\Gamma = \gamma^1$ in (3.1) and using (3.3) we find

$$\langle j^1(x) \rangle = iN \int \frac{d\omega}{2\pi} [A(x) - D(x)]. \quad (3.5)$$

But we have already proved that $A(x) = D(x)$ in *any* static background $(\sigma(x), \pi(x))$ (Eq.(2.24)). Thus, each frequency component of $\langle j^1 \rangle$ vanishes separately, and (3.4) holds identically. It is remarkable that the generalized supersymmetry of the Dirac operator guarantees the consistency of any static $(\sigma(x), \pi(x))$ background.

Expressions for other bilinear condensates may be derived in a similar manner (here we write the unsubtracted quantities). Thus, substituting $\Gamma = \gamma^0$ in (3.1) and using (3.3), (2.24) and (2.28), we find that the fermion density is

$$\langle j^0(x) \rangle = iN \int \frac{d\omega}{2\pi} [B(x) - C(x)] = iN \int \frac{d\omega}{2\pi} \frac{2\omega B(x) - i\partial_x D(x)}{\omega + \pi(x)}. \quad (3.6)$$

Similarly, the scalar and pseudoscalar condensates are

$$\langle \bar{\psi}(x)\psi(x) \rangle = N \int \frac{d\omega}{2\pi} [A(x) + D(x)] = 2N \int \frac{d\omega}{2\pi} D(x), \quad (3.7)$$

and

$$\langle \bar{\psi}(x)\gamma^5\psi(x) \rangle = N \int \frac{d\omega}{2\pi} [B(x) + C(x)] = N \int \frac{d\omega}{2\pi} \frac{2\pi(x)B(x) + i\partial_x D(x)}{\omega + \pi(x)}. \quad (3.8)$$

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